

Fast Computation of the Arnold Complexity of Length 2^n Binary Words

Yuri V. Merekin

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Sobolev Institute of Mathematics, SB RAS, Novosibirsk 630090, Russia.

E-mail: merekin@math.nsc.ru

Abstract. For fast computation of the Arnold complexity of length 2^n binary words we obtain an upper bound for the Shannon function $Sh(n)$.

Keywords: binary word; word complexity; Arnold complexity; Shannon function.

1 Introduction

Analyzing word complexity usually involves studying the fragments of a word or the process of its construction (see [2] for instance). Arnold introduced [1] a new concept of complexity of a word. The measure of this complexity is determined by the “stability” of a word under the iterated action of a certain operator.

Consider an arbitrary nonperiodic binary word $w = x_1x_2 \dots x_{2^n}$, with $w \neq v^k$ and $k \geq 2$, of length $|w| = 2^n$ for $n \geq 1$. Denote by (w) the infinite periodic word $(w) = ww \dots$. Henceforth, by a “word (w) ” we understand “an infinite periodic word (w) ”. Consider the scheme (word chain)

$$(w) = (w_1), (w_2), \dots, (w_s) = (v), \quad (1)$$

in which the first word is arbitrary, and every word $(w_i) = y_1y_2 \dots$ generates the next word $(w_{i+1}) = z_1z_2 \dots$, for $1 \leq i \leq s-1$, using the operator

$$F(\cdot, h_i) : (w_i) \mapsto (w_{i+1}) : z_j = y_j \oplus y_{j+h_i}, \quad (2)$$

where $j \geq 1$, $1 \leq h_i = 2^{n_i}$, $0 \leq n_i \leq n$, and \oplus stands for modulo 2 addition; thus, $F(w_i, h_i) = (w_{i+1})$. The number h_i is called the *rank* of the operator in (2), and the number s , the *length* of the scheme (1).

Denote by $S(w, v, h, s)$ the type of schemes with the first word (w) , the last word (v) , the maximal rank h of operators involved, and the scheme length s . For every (w)

there exists a minimal s such that all words in the scheme $S(w, 0, 1, s)$ are distinct, and $F(v, 1) = (0)$. A scheme of this type is called a *complexity scheme*. Every word (w) has a unique complexity scheme. The number $s - 1$ is called the *complexity* of the binary word (w) and is denoted by $A(w)$. Arnold introduced [1] the concept of complexity of a binary word in a more general form, which coincides with our definition of complexity when the word length equals 2^n . The complexity of a periodic word (w) is equal to the complexity of the finite word w .

In an arbitrary scheme $S(w, v, 1, s)$ select a word chain

$$(w) = (w_1), (w_{1+l_1}), \dots, (w_{1+l_1+\dots+l_t}) = (v), \quad (3)$$

where $l_i \geq 1$ for $1 \leq i \leq t \leq s - 1$. If in (3) each word $(w_{1+l_1+\dots+l_i})$ for $1 \leq i \leq s - 1$, coincides with $F(w_{1+l_1+\dots+l_{i-1}}, h_i)$, $h_i = l_i$, then the word chain in (3) is a scheme of type $S(w, v, h, s_t)$ with $s_t = 1 + t$, which is called *equivalent* to $S(w, v, 1, s)$.

In [3] we proved

Theorem 1.1 *Every scheme $S(w, v, 1, 2^n + 1)$ with $n \geq 0$ is equivalent to the elementary scheme $S(w, v, 2^n, 2)$.*

In a word $w = x_1 x_2 \dots x_{2^n}$, $n \geq 1$, select 2^{n-m} positions, with $0 \leq m \leq n - 1$, such that the distances between two neighboring selected positions are the same and equal to 2^m . Using the selected positions, form the word $u = x_i x_{i+2^m} \dots x_{i+2^{n-m}-2^m}$ of length 2^{n-m} . Denote the infinite word $(u) = uu \dots$ by $(w)_{2^{n-m}}^{x_i}$ and call it a *thinned-out word*. The number 2^{n-m} is called the *step* of the thinned-out word $(w)_{2^{n-m}}^{x_i}$. Observe that every thinned-out word is a linearly ordered set of indices of positions of w . The length of the period of $(w)_{2^{n-m}}^{x_i}$ can be less than 2^{n-m} . For $m = 0$ we have $(w) = (w)_{2^n}^{x_1}$.

Given a word (w) , for a fixed value of m there exist 2^m different thinned-out words

$$(w)_{2^{n-m}}^{x_1}, (w)_{2^{n-m}}^{x_2}, \dots, (w)_{2^{n-m}}^{x_{2^m}}, \quad n \geq 1, \quad 0 \leq m \leq n - 1. \quad (4)$$

For instance, for $n = 3$ we have $(w)_{2^3}^{x_1} = (x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8)$, $m = 0$;

$(w)_{2^{3-1}}^{x_1} = (x_1 x_3 x_5 x_7)$, $(w)_{2^{3-1}}^{x_2} = (x_2 x_4 x_6 x_8)$, $m = 1$;

$(w)_{2^{3-2}}^{x_1} = (x_1 x_5)$, $(w)_{2^{3-2}}^{x_2} = (x_2 x_6)$, $(w)_{2^{3-2}}^{x_3} = (x_3 x_7)$,

$(w)_{2^{3-2}}^{x_4} = (x_4 x_8)$, $m = 2$.

Define the operation of taking the union of thinned-out words, denoted by the symbol $*$. The definition of a thinned-out word implies that each of the positions x_1, x_2, \dots, x_{2^n} appears in the thinned-out words (4) exactly once since it is the union of arithmetic progressions with differences equal to powers of 2. Thus, we can express (w) as

$$(w) = (w)_{2^{n-m}}^{x_1} * (w)_{2^{n-m}}^{x_2} * \dots * (w)_{2^{n-m}}^{x_{2^m}} \quad n \geq 1, \quad 0 \leq m \leq n - 1.$$

We group the words into two sorts: even words and odd words. The word $(w) = ww \dots$, where $w = x_1 x_2 \dots x_{2^n}$ with $x_i \in \{0, 1\}$ and $1 \leq i \leq 2^n$ for $n \geq 0$, is called *even* whenever $x_1 \oplus x_2 \oplus \dots \oplus x_{2^n} = 0$, and *odd* whenever $x_1 \oplus x_2 \oplus \dots \oplus x_{2^n} = 1$. For calculating the parity of the thinned-out words $(w)_{2^{n-m}}^{x_i}$, where $|w| = 2^n$ with $n \geq 1$

and $0 \leq m \leq n-1$, of a word $(x_1 x_2 \dots x_{2^n})$, we gave a simple algorithm [4], which uses modulo 2 addition $2^n - 1$ times, and proved

Theorem 1.2 *For every binary word (w) the length of whose period is equal to 2^n , $n \geq 1$, all thinned-out words $(w)_{2^{n-m}}^{x_i}$ for $0 \leq m \leq n-1$ and $1 \leq i \leq 2^m$ are odd if and only if the complexity of (w) is equal to $A(w) = 2^n - 2^m + 1$.*

Express the complexity $A(w)$ of an arbitrary word (w) with $|w| = 2^n$ for $n \geq 1$ as

$$A(w) = a_{n-1}2^{(n-1)} + a_{n-2}2^{(n-1)-1} + \dots + a_02^{(n-1)-(n-1)}, \quad (5)$$

or the binary number $a_{n-1}a_{n-2}\dots a_0$, where $a_i \in \{0, 1\}$ for $0 \leq i \leq n-1$.

Express the complexities $A(w)$, which according to Theorem 1.2 we calculate by finding the parities of thinned-out words, as the binary numbers

$$\begin{aligned} 2^n - 2^1 + 1 &= a_{n-1}a_{n-2}\dots a_0 = 11\dots 111, \\ 2^n - 2^2 + 1 &= a_{n-1}a_{n-2}\dots a_0 = 11\dots 101, \dots, \\ 2^n - 2^{n-1} + 1 &= a_{n-1}a_{n-2}\dots a_0 = 10\dots 001, \\ 2^n - 2^{n-1} + 0 &= a_{n-1}a_{n-2}\dots a_0 = 10\dots 000, \quad n \geq 1, \end{aligned} \quad (6)$$

where all numbers are odd with the exception of $2^n - 2^{n-1}$.

Refer to a word (v) as *final* if $A(v)$ equals one of the values in (6). Every complexity scheme $S(w, 0, 1, s)$ with $s = 2^n$ for $n \geq 1$ contains $n + (n-1) + \dots + 1$ final words.

Using $t \geq 1$ operators (2) of ranks h_1, h_2, \dots, h_t , transform the complexity scheme $S(w, 0, 1, s)$ into a scheme

$$(w) = (w_1), (w_{1+h_1}), \dots, (w_{1+h_1+\dots+h_t}) = (v), (w_{1+h_1+\dots+h_t+1}), \dots, (0)$$

with the final word (v) . Then

$$A(w) = h_1 + h_2 + \dots + h_t + A(v), \quad (7)$$

where $A(v)$ is one of the numbers (6). It is obvious that in order to transform (w) into (v) , every permutation of the ranks h_1, h_2, \dots, h_t of the operators $F(\cdot, h_i)$ for $1 \leq i \leq t$ is admissible.

2 Shannon Function

Refer to the minimal number of operators $F(\cdot, h_i)$ required to transform (w) into one of the final words (v) as the *complexity of transformation* of (w) into (v) , and denote it by $A(w, v) = \min_{w \rightarrow v} A(w)$.

Our goal is to find the Shannon function $\max_w \min_{w \rightarrow v} A(w)$, which we denote by $Sh(n)$.

Consider an example. Take a complexity scheme $(w_{16}), (w_{15}), \dots, (w_2), (w_1), (w_0)$. For the words (w) with $|w| = 2^n$ for $0 \leq n \leq 4$ five values of complexity exist, for

which we have expressions as in (7). In each of these cases an operator $F(w_i, 2^r) : (w_i) \mapsto (w_j)$ is used only once. Table 1 presents the results of calculating $A(w)$ for all words (w) with $|w| = 2^n$ for $1 \leq n \leq 4$.

Table 1

(w_i)	$A(w_i)$	$F(w_i, 2^r) : (w_i) \mapsto (w_j)$	$A(w_i) = 2^r + A(w_j)$
(w_{16})	$2^4 - 2^0 + 1$		
(w_{15})	$2^4 - 2^1 + 1$		
(w_{14})		$F(w_{14}, 2^0) : (w_{14}) \mapsto (w_{13})$	$A(w_{14}) = 2^0 + (2^4 - 2^2 + 1)$
(w_{13})	$2^4 - 2^2 + 1$		
(w_{12})		$F(w_{12}, 2^2) : (w_{12}) \mapsto (w_8)$	$A(w_{12}) = 2^2 + (2^3 - 2^0 + 1)$
(w_{11})		$F(w_{11}, 2^1) : (w_{11}) \mapsto (w_9)$	$A(w_{11}) = 2^1 + (2^4 - 2^3 + 1)$
(w_{10})		$F(w_{10}, 2^0) : (w_{10}) \mapsto (w_9)$	$A(w_{10}) = 2^0 + (2^4 - 2^3 + 1)$
(w_9)	$2^4 - 2^3 + 1$		
(w_8)	$2^3 - 2^0 + 1$		
(w_7)	$2^3 - 2^1 + 1$		
(w_6)		$F(w_6, 2^0) : (w_6) \mapsto (w_5)$	$A(w_6) = 2^0 + (2^3 - 2^2 + 1)$
(w_5)	$2^3 - 2^2 + 1$		
(w_4)	$2^2 - 2^0 + 1$		
(w_3)	$2^2 - 2^1 + 1$		
(w_2)	$2^1 - 2^0 + 1$		
(w_1)	2^0		

In the next theorem we consider the general case for $n \geq 5$.

Theorem 2.1 *Given a word (w) with $|w| = 2^n$ for $n \geq 5$, we have*

$$Sh(n) \leq \begin{cases} \lfloor n - 2\sqrt{n} + 1 \rfloor & \text{when the binary number } A(w) \text{ is odd,} \\ \lfloor n - 2\sqrt{n} + 2 \rfloor & \text{otherwise.} \end{cases}$$

Proof. Case 1. Assume that the value of the complexity $A(w)$ is odd.

Fix $A(w)$ and estimate the minimal number of operators transforming (w) into a final word (v) . It is obvious that in this case the ranks of all operators are distinct.

In order to estimate $A(w, v)$, consider the result of the action of the operator of (2) on the coefficients $a_{n-1}, a_{n-2}, \dots, a_0$ of (5). if

$$F(u_1, h = 2^{(n-1)-i}) = (u_2), \quad 0 \leq i \leq n-1,$$

then $A(u_1) - A(u_2) = 2^{(n-1)-i}$. Moreover, two variants are possible for changing the values of $a_{n-1}, a_{n-2}, \dots, a_0$:

$$(a_{(n-1)-i} = 1) \mapsto (a_{(n-1)-i} = 0); \quad (8)$$

$$\begin{aligned}
(a_{(n-1)-i+j} = 1) &\mapsto (a_{(n-1)-i+j} = 0), \\
(a_{(n-1)-i+(j-1)} = 0) &\mapsto (a_{(n-1)-i+(j-1)} = 1), \dots, \\
(a_{(n-1)-i} = 0) &\mapsto (a_{(n-1)-i} = 1),
\end{aligned} \tag{9}$$

where $1 \leq i \leq n-1$, $1 \leq j \leq i$.

In case (8) the rank of $h = 2^{(n-1)-i}$ coincides with one of the terms in the sum (5). The action of the operator removes the term $2^{(n-1)-i}$ from (5). For instance, the operator of rank $h = 2^1$ transforms $A(w) = 2^4 + 2^3 + 2^1 + 2^0$ into $2^4 + 2^3 + 2^0$.

In case (9), when the rank of $h = 2^{(n-1)-i}$ is distinct from all terms of (5), we remove the term $2^{(n-1)-i+j}$ with minimal j . Simultaneously, we add to (5) the terms

$$2^{(n-1)-i+(j-1)}, 2^{(n-1)-i+(j-2)}, \dots, 2^{(n-1)-i}. \tag{10}$$

For instance, the operator of rank $h = 2^1$ transforms $A(w) = 2^4 + 2^2 + 2^0$ into $2^4 + 2^1 + 2^0$.

Consider the case when we can apply (9) in order to calculate $A(w, v)$.

Suppose that (5) includes a run $a_i = a_{i-1} = \dots = a_{i-l+1} = 1$ of neighboring unit coefficients of maximal length, where $i \leq n-1$ and $i+l-1 \geq 1$, which we denote by $s(i, l)$. Several runs of maximal length may exist; for instance, $A(w) = 2^5 + 2^4 + 2^2 + 2^1 + 2^0$ includes two such runs: $s(5, 2)$ and $s(2, 2)$.

For a fixed value $A(w)$ of complexity choose a run $s(i, l)$ arbitrarily. If $A(w)$ is distinct from (6) then the sum in (5), in addition to the l terms $2^i, 2^{i-1}, \dots, 2^{i-l+1}$ and the term 2^0 , also involves t distinct terms with $1 \leq t \leq n-l-2$. Once we remove these t terms, the remaining sum would coincide with one of the sums in (6).

To remove t distinct terms from (5) using (8) we need t operators $F(\cdot, h_i)$, of distinct ranks h_1, h_2, \dots, h_t . For instance, in $A(w) = 2^5 + 2^4 + 2^2 + 2^1 + 2^0$ choose a run of neighboring unit coefficients of maximal length $s(5, 2)$ and remove the terms 2^2 and 2^1 . This yields the sum $2^5 + 2^4 + 2^0$, which coincides with one of the sums in (6).

The transformation process $A(w) \mapsto A(v)$ involves a unique case when the replacement of the variant (8) by the variant (9), in which the number of terms increases, fails to increase the number $F(\cdot, h_i)$ of operators in the transformation $A(w) \mapsto A(v)$. Moreover, the form of the final word changes: it additionally includes all terms of (10). This happens when in $A(w) = 2^j + 2^i + \dots + 2^0$ with $j \geq i+2$ we choose a run $s(i, l)$ and apply the operator $F(w, h = 2^{i+1})$. Then we remove from $A(w)$ the term 2^j and transform the run $s(i, l)$ into the run $s(j-1, l+(j-i-1))$. For instance, for $A(w) = 2^5 + 2^3 + 2^2 + 2^0$ choose the run $s(3, 2)$. Then the operator $F(w, h = 2^4)$ transforms $A(w) = 2^5 + 2^3 + 2^2 + 2^0$ into $2^4 + 2^3 + 2^2 + 2^0$, while the run $s(3, 2)$ goes into $s(4, 3)$.

Consequently, for removing t distinct terms from (5) the application of (9) is not necessary, and for finding $A(w, v)$ we may use only the operators resulting in (8).

Remark Every odd binary number $A(w)$ includes the term 2^0 , which we do not remove while constructing $A(w, v)$. Consequently, the operator of rank $h = 1$ is not used while obtaining $A(w, v)$.

Denote by $\nu(N)$ the number of 1's in the binary expression for a nonnegative integer N . The arguments above imply that

$$A(w, v) = \min_{w \rightarrow v} A(w) = \nu(A(w)) - l - 1,$$

where l is the length of the maximal run $s(i, l)$.

Let us find $Sh(n)$ for nonfinal words (w) with $|w| = 2^n$ for $n \geq 1$.

Construct a continuous function which, copying the process of removal of the maximal number of 1's from a binary number $A(w)$, determines an upper bound for $Sh(n)$.

Divide a line of integer length $n \geq 4$ into x segments, with $2 \leq x \leq n/2$, of the same length n/x . Keeping one of the segments intact, remove the beginning of all other segments to leave only a finite part of unit length. Then the total length of the removed segments is estimated by the convex function

$$f(x) = (x-1)(n/x-1),$$

which has one extremum. Find the derivative $f'(x)$ and set it equal to zero:

$$f'(x) = n/x^2 - 1 = 0.$$

This yields $x = \sqrt{n}$ and the maximal value attained by the function $f(x)$, equal to

$$f(x = \sqrt{n}) = n - 2\sqrt{n} + 1.$$

For odd binary numbers $A(w)$ Theorem is proved.

Case 2. Assume that the value of the complexity $A(w)$ is even. Estimate the minimal number of operators (2) required for calculating $A(w, v)$.

Suppose that the length n binary number $A(w)$ includes $\nu(A(w))$ digits 1, where $2 \leq \nu(A(w)) \leq n-1$, and

$$a_{(n-1)-j} = 1, a_{(n-1)-j-1} = 0, \dots, a_0 = 0, \quad 1 \leq j \leq n-2. \quad (11)$$

Two variants for calculating $A(w, v)$ are possible.

Subcase 2.1. From the binary number $A(w) = a_{n-1}a_{n-2}\dots a_0$, which contains $\nu(A(w))$ digits 1, remove $\nu(A(w)) - 1$ digits 1 using (8). This yields a binary number $A(v)$ with a unique digit 1. The number of operators $A(w) \mapsto A(v)$ equals

$$\nu(A(w)) - 1. \quad (12)$$

Subcase 2.2. Apply the operator $F(w = w_1, h = 1): (w_1) \mapsto (w_2)$ once. As a result, the even number $A(w_1)$ goes into the odd number $A(w_2)$, and $A(w_2) = A(w_1) - 1$. Carry out further calculations according to the algorithm of case 1, in which by Remark 2.2 the operator (2) of rank 1 is not used.

Upon the application of $F(w = w_1, h = 1)$ to the number $A(w)$ all binary digits in (11) switch their values in accordance with (9). Therefore, the number $A(w_2)$ includes

$$\nu(A(w_2)) = \nu(A(w)) - 1 + j$$

digits 1. Removing from $A(w_2)$ all digits 1 except for l of those in $s(i, l)$ and $a_0 = 1$, we obtain $A(v)$ with $\nu(A(v)) = l + 1$ digits 1. The number of operators transforming $A(w)$ into $A(v)$ equals

$$\nu(A(w_2)) - \nu(A(v)) + 1 = \nu(A(w)) + j - l - 1. \quad (13)$$

In order to estimate the complexity

$$A(w, v) = \min_{w \rightarrow v} A(w)$$

we choose the variant with the minimal number of operators. A comparison of (12) and (13) shows that this number occurs in subcase 2.1 for $j \geq l$ and in subcase 2.2 for $j \leq l$.

For instance, for $A(w) = 101110100$ we choose subcase 2.2:

$$\begin{aligned} A(w) &= 101110100, & \nu(A(w)) &= 5, & j &= 2; \\ A(w_2) &= 101110011, & \nu(A(w_2)) - 1 + j &= 6, & l &= 3; \\ A(v) &= 001110001, & \nu(A(v)) &= l + 1 = 4; \\ A(w) \mapsto A(v) &\implies 1 \quad 11, & \nu(A(w)) + j - l - 1 &= 3. \end{aligned}$$

Observe that if the operator $F(w = w_1, h = 1) : (w_1) \mapsto (w_2)$ in subcase 2.2 generates an odd word (w_2) , for which we have already established the estimate $\lfloor n - 2\sqrt{n} + 1 \rfloor$, then for even $A(w_1)$ we have the estimate $\lfloor n - 2\sqrt{n} + 2 \rfloor$ since $A(w_1) = A(w_2) + 1$.

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